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A possible theory of partial differential equations

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Abstract. The current gold standard for solving [nonlinear] partial differential equations, or [N]PDEs, is the simplest equation method, or SEM. Another prior technique for solving such equations, the G'/G-expansion method, appears to branch from the simplest equation method (SEM). This study discusses a new method for solving PDEs called the generating function technique (GFT) which may establish new precedence concerning SEM. First, the study shows how GFT relates to SEM and the G'/G-expansion method. Next, the paper describes a new theorem that incorporates GFT and Ring theory in the finding of solutions to PDEs. Then the novel technique is applied in the derivation of new or exotic solutions to the Benjamin-Ono, a QFT (nonlinear Klein-Gordon), and Good Boussinesq-like equations. Finally, the study concludes via a discourse on the reasons why the technique is better than SEM and G'/G-expansion method and the scope and range of what GFT could accomplish in the realm of mathematics, specifically differential equations.

Keywords: differential equations, generating function technique, G'/G-expansion method, simplest equation method

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Возможная теория дифференциальных уравнений с частными производными

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Аннотация. На данный момент метод простейшего уравнения является стандартным методом для решения [нелинейных] уравнений в частных производных. Другим преимущественным методом решения таких уравнений является метод G'/G-разложения, который является ответвлением от метода простейших уравнений. В данной работе рассматривается новый метод решения уравнений в частных производных – метод производящих функций, который может стать приоритетным в отношении метода простейших уравнений. Исследование показывает, как метод генерирующих функций соотносится с методом простейших уравнений и методом G'/G-разложения. Описывается новая теорема, которая включает технику производящих функций и теорию кольца для поиска решений уравнений в частных производных. Нестандартная техника применяется для вывода новых или необычных решений уравнений Бенджамина – Оно, нелинейных уравнений Клейна – Гордона и уравнений Буссинеска. И, наконец, обсуждаются причины, по которым метод производящих функций лучше, чем методы простейших уравнений и G'/G-разложения, а также каких высот можно достигнуть в области математики, в частности дифференциальных уравнений, благодаря этому методу.

Ключевые слова: дифференциальные уравнения, метод производящих функций, метод G'/G-разложения, метод простейшего уравнения

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1. Introduction

Many notable mathematicians, like Lawrence Evans, suggest a general theory of [nonlinear] partial differential equations cannot exist. He claims there can never be a pithy theory to describe partial differential equations due to its vast number of [diverse] sources [1]. However, there are semi-analytical methods, like Adomian decomposition and homotopy analysis, which have been shown to solve a large variety of [N]PDEs [2, 3]. Unfortunately, these techniques are not purely analytical, come with extremely high computational costs, and are very time-consuming. Therefore, one must truly ask can one find or erect a purely analytical method for solving partial differential equations, especially [N]PDEs?

Over the last couple of decades, several analytical methods, which required some degree of computation, for solving some [N]PDEs had been developed. In a groundbreaking paper in 2005, Kudryashov established a rapid computational method for finding exact solutions to NPDEs: this technique was known as the simplest equation method [4]. The following year, He and Wu unveiled a technique, called the exponential function method, which could solve a large variety of [N]PDEs also by computer [5]. In 2008, Wang had devised another computational technique for solving [N]PDEs called the G'/G-expansion method [6]. Over the next decade and a half, variations from the above and other fewer notable techniques were introduced to the world of mathematics.

Stone-Weierstrass theorem states that a continuous function can be closely approximated to a polynomial [7]. Assuming the polynomial is a formal power series of at least an exponential function, it should converge to the exact solution of partial differential equations with the right coefficients [8]. If one wishes to devise a method that can solve a wide variety of partial differential equations, (s)he may have to heed this theorem and use a formal power series of an exponential function with the proper coefficients [9].

The above paragraph points to a method for solving [N]PDE that is well-grounded in aspects of Ring theory. This theory is the study of algebraic structures, called rings, in which multiplication and addition are well defined and are like operations associated with integers [10]. There are many types of rings, like commutative and noncommutative, and its theory delves into their structures, representations, etc. For example, a ring of a formal power series is the set of all formal power series in X with coefficients in a commutative ring R [11].

Also, the method for solving [N]PDEs gave credence to generating functions, an instrument of combinatorics, as being a putative valuable utility. Initially, Abraham de Moivre developed generating function to derive solutions to linear recurrence problems [12]. Over time, many individuals used formal power series for defining numerical sequences via generating functions [12]. Also, this author came to believe generating functions may serve as a perfect vessel or analytical means for solving [N]PDEs.

A technique, called the Generating Function[s] Technique (GFT), for solving at least homogeneous [N]PDEs will be discussed in this paper. First, the paper will show how the method incorporates a set of Laurent series of formal power series with a solution, derived from an auxiliary/characteristic equation, and trigonometric-based coefficients; thus, the paper will compare GFT to other methods (i.e., the simplest equation (SEM), G'/G-expansion methods). Next, the study will show how the set of formal power series, hence general and exact solution to a [N]PDE is connected to polynomial rings via a theorem: this theorem is primarily supported by aspects of Ring theory. Then the paper will apply the theorem on a few examples of [N]PDEs to find new or exotic solutions. Finally, the study will conclude with a more exquisite explanation on why the method is more highly effective in comparison to other techniques and what other functions GFT can perform.

2. Methodology

The relationship between generating functions and the solution to the Riccati equation

The Riccati equation, a first-order ordinary differential equation (ODE), is the following expression:

$$\phi'(\xi)+\phi(\xi)^2+\phi(\xi)=0,$$

where ϕ is the solution to the equation and ξ is the [transformed] variable [13]. Solution ϕ is defined as:

$$\phi(\xi)=\frac{1}{e^{\xi-c_1}-1}.$$

Now consider a generating function γ, or:

$$\gamma(\xi)=\sum_{i=0}^{\infty}p_i f(\xi)^i,$$

where f is some function in terms of ξ and p_i is the i-th parameter or coefficient in the formal power series [12]. If one lets function f equal to $e^{\frac{\xi}{2}}$ and parameter p_i equal the Lucas L$_i$ combinatorial number about zero divided by two, or $\cos^2\left(\frac{i\pi}{2}\right)$, the generating function γ becomes:

$$\gamma(\xi)=\sum_{i=0}^{\infty}L_i(\xi)e^{\frac{i(\xi-c_1)}{2}},$$

or

$$\gamma(\xi)=\frac{1}{1-e^{\xi-c_1}}.$$

It is noteworthy to state γ is equal to negative ϕ. In other words, the solution to the Riccati equation can be redefined as a generating function.

The relationship of other quintessential expressions and generating functions

There are other important functions used to solve [nonlinear] PDEs that can be defined as generating functions. The table below provides a list of relationships between generating functions and quintessential expressions utilized in solving [nonlinear] PDEs.

$\gamma(\xi)$	$f(\xi)$	p_i
$\frac{1}{1-e^{\xi-c_1}}$	$e^{\xi-c_1}$	$2B_i(0)$ $Le_i(0)$
$\frac{1}{1-e^{\xi-c_1}}$	$e^{\frac{(\xi-c_1)}{2}}$	$L_i(0)$
$\frac{1}{e^{\xi-c_1}+1}$	$e^{\frac{(\xi-c_1)}{2}}$	$2U_i(0)$
$\frac{1}{\cosh(\xi-c_1)-1}$	$e^{\xi-c_1}$	$2H_i(0)$
$-\operatorname{csch}\left(\frac{(\xi-c_1)}{2}\right)$	$e^{\frac{(\xi-c_1)}{2}}$	$2F_i(0)$
$\operatorname{sech}\left(\frac{(\xi-c_1)}{2}\right)$	$e^{\frac{(\xi-c_1)}{2}}$	$2\sqrt{F_i(0)}$

B_i, Le_i, L_i, U_i, F_i, and H_i are the i-th binomial, Laguerre L, Lucas L, Chebyshev U, Fibonacci, and Harmonic [combinatorial] numbers about zero, respectively.

The general solution associated with GFT

In general, consider the following expression:

$$p_s \geq \left|\frac{(d_l-d_n)}{(p_n-p_l)}-n_n+1\right|,$$

where p_s is the power of the solution u in a putative series or the level of Laurent serial truncation for solution u, d_l is the highest degree of the linear terms, d_n is the total degree of a nonlinear term, p_n is the total power of the same nonlinear term, p_l is the highest power of the linear term, which is one, and n_n is the number of basic nonlinear terms (including the source type). Note: finding the exact p_s is not necessary but it will lower the time necessary for deriving the solution u.

SEM defines the general solution of a [nonlinear] PDE as a rudimentary linear combination or simple sum of the solution to the Riccati equation, or:

$$U_i(\xi)=\sum_{i=0}^{p_s}\eta_i\phi(\xi)^i,$$

where η_i is the i-th coefficient or parameter [4]. The Riccati equation serves as an auxiliary equation to SEM and more specifically the G'/G-expansion method [4, 6].

Now consider the [transformed] general solution for GFT which involves a [truncated] Laurent series [14]. The putative [transformed] general solutions $u(x)$, or $U(\xi)$, too many PDEs is defined as:

$$U(\xi)=\sum_{i=1}^{2}\left[\sum_{j=-p_s}^{p_s}\left(a_{ij}\left(\sum_{k=0}^{\infty}2f(\xi)^k F_k(0)^i\right)^j+b_{ij}\left(\sum_{k=0}^{\infty}2f(\xi)^k C_k(0)^i\right)^j\right)\right],$$

where the (square root of the) Fibonacci k-th number/parameter given/for zero is the following expression:

$$F_k(\xi)=\sin\left(\frac{\pi k}{2}\right),$$

and the Chebyshev U k-th number/parameter given/for zero is expressed as:

$$C_k(\xi)=\cos\left(\frac{\pi k}{2}\right).$$

Note: the ansatz transformed variable ξ is a linear array of intermediates/variables, or the following expression:

$$\xi=\alpha t+\beta x,$$

where α and β are coefficients to the variables or intermediates t and x, respectively. This expression is only for a 1 + 1-dimensional system.

If one wishes not to work with coefficients with negative indices, then shift the [truncated] Laurent series via p_s, like:

$$U(\xi)=\sum_{i=1}^{2}\left(\sum_{j=0}^{2p_s}\left(a_{ij}\left(\sum_{k=0}^{\infty}2f(\xi)^k F_k(0)^i\right)^{j-p_s}+b_{ij}\left(\sum_{k=0}^{\infty}2f(\xi)^k C_k(0)^i\right)^{j-p_s}\right)\right).$$

This expression/transformed general solution involves an offset. Through GFT, the auxiliary/characteristic equation used for the facilitation of SEM and the G'/G-expansion method is a basic first-order ODE, or:

$$f'(\xi)+f(\xi)=0.$$

Its solution is simply defined as:

$$f(\xi)=c_1e^{-\xi}.$$

Using the solution to the above basic auxiliary equation in the general solution to some principal partial differential equation gives rise to hyperbolic secant, hyperbolic cosecant, hyperbolic sine, hyperbolic cosine via Fibonacci or sine-based parameters/generating functions and expressions involving one plus hyperbolic tangent and cotangent via Chebyshev U or cosine-based parameters/generating functions raised by various powers.

The degree of "diversity" of solutions u of [nonlinear] PDEs established by GFT will be dependent upon the complexity of the auxiliary equation used. The auxiliary equation of GFT, which will be used to derive f, hence generating function χ, can be any order linear ODE.

3. Theorem

Let u_g be the general solution while u_e be the exact solution to the differential equation F. The differential equation F is defined as:

$$F(u,u_t,u_x,u_{tt},u_{xx},u_{xt},...) = 0.$$

Definition 3.1. *The general solution u_g, which is a set of formal power series and their multiplicative inverses, is a ring formed from the set of polynomials in one or more indeterminates with coefficients in another ring/field, or $u_g \in R[[x]]^{\{E\}}$. The general solution u_g may also include [hyperbolic] trigonometric functions (i.e. hyperbolic secant, hyperbolic cosecant, etc.) raised by various powers which are generally polynomial ring analogs.*

***Definition 3.2**. Transformed general solution U_g, which is a set of formal power series and their multiplicative inverses, is a ring formed from the set of polynomials with one indeterminate with coefficients in another ring/field, or $U_g \in R[[f_1(\xi), f_2(\xi), ..., f_m(\xi)]]^{\{E\}}$, where $f_i(x)$ is equal to the product of some i-th constant and the transformed variable ξ and $1 \le i \le m$. The transformed general solution U_g may also include [hyperbolic] functions raised by various powers which are polynomial ring analogs.*

A formal power series (of an exponential function) establishes a polynomial ring $R[[\boldsymbol{x}]]^{\{E\}}$ [15, 16, 17]. (Note: {E} designates an exponentiated entity.) On the other hand, the Maclaurin/Taylor series establishes a polynomial ring analog [17]. The multiplicative inverse of some formal power series will produce either another formal power series or the analog of a polynomial ring $R[[\boldsymbol{x}]]^{\{E\}}$. Since polynomial rings can be commutative and associative or undergo both addition and multiplication, another polynomial ring [analog] is generated by raising the power of formal power and Maclaurin/Taylor series. The general solution u_g and its transformed general solution U_g are a set of Laurent series of polynomial rings and their analogs. Since polynomial rings can be commutative and associative, their net sum is a larger polynomial ring [analog].

Lemma 3.3. *If U_g is a polynomial ring, then the transformed differential equation F will be a polynomial ring also. In other words, $F \in R[[f_1(\xi), f_2(\xi), ..., f_m(\xi)]]^{\{E\}}$.*

Lemma 3.4. *If the common denominator C of transformed differential equation F is a polynomial ring, or $C \in R[[\xi]]^{\{E\}}$, then the product of the common denominator C and transformed differential equation F is another polynomial ring P, or $C\ X\ F = P \in R[[f_1(\xi), f_2(\xi), ..., f_m(\xi)]]^{\{E\}}$.*

Lemma 3.5. *If the polynomial ring P possesses features of an ideal, such as generators (i.e. $\langle e^{\xi} \rangle$, etc.), it could be viewed as being a free ideal ring, or fir.*

Lemmas 3.3 to 3.5 states that one can establish a group of coefficients and (set of) generator[s] via the free ideal ring *P* after plugging the transformed general solution U_g into the differential equation *F* and multiplying the result by its common denominator. The transformed general solution U_g is operated on by a composition of operators in the differential equation which forms a differential polynomial ring *F* [18]. This ring can be viewed as a dividing [formal] power series. By multiplying the differential polynomial ring *F* by its common denominator series *C*, an individual would be left with the numerator series which is another polynomial ring *P* [19]. The most latter polynomial ring is comprised of free modules, which have a non-zero ring of coefficients and linearly independent generating set (generator[s]), or:

$$e^{f_i(\xi)} = \left\{1, e^{f_i(\xi)}, e^{2f_i(\xi)}, \ldots, e^{nf_i(\xi)}\right\},$$

where the [exponentiated] function $f_i(\xi)$ is the *i*-th generator. Since the product polynomial *P* possesses non-zero coefficients with multiplicative inverses and a [unique] rank linked to the generating set, it can also be viewed as being a free ideal ring [20].

The generating set is derived from the solution of the auxiliary/characteristic equation, utilized in the establishment of the transformed general solution U_g, and used to define the rank of free ideal ring *P*. For instance, if the solution to the auxiliary/characteristic equation is:

$$f(\xi) = c_1 e^{-\xi},$$

therefore, the generator is $\langle e^{\xi} \rangle$, where $f_1(\xi)$ is equal to the product of transformed variable ξ and the value 1. The free ideal ring *P* is established by this sole generator and has a rank equal to 1. On the other hand, the auxiliary/characteristic solution is defined as follows:

$$\phi(\xi) = c_1 \cos(\xi) + c_2 \sin(\xi),$$

then the generator is $\langle e^{i\xi} \rangle$, where $f_1(\xi)$ is equal to the product of transformed variable ξ and the imaginary unit ***i***. The free ideal ring *P* established by this lone generator also has a rank equal to 1. In other instances, the solution to the auxiliary/characteristic equation can produce *m* generators, thus the resultant product polynomial ring *P* generally yields a rank equal to *m* [20]. (Note: if the number of independent columns *n*, or special rank, is greater than or equal to the *m*, or general rank, then the [right] free ideal ring is a *n-fir* or *semifir*, respectively. Also, the major difference between a division and a free ideal ring is their level of symmetry: division rings are ALL left-right symmetric while free ideal rings are not.)

Lemma 3.6. *If the coefficients of the polynomial free ideal ring P are made to equal to zero, then the exact solution* u_e *may exist.*

Definition 3.7. *The exact solution* u_e *is a polynomial subring of the transformed general solution* U_g.

Lemma 3.6 and definition 3.7 claim that an individual can derive possibly at least one exact solution u_e from the transformed general solution U_g by interrogating the coefficients linked to the free ideal ring P. The coefficients associated with the generator[s], linked to the free ideal ring P, form algebraic equations that should be set to zero. Thus, an individual considers the free ideal ring P to be a nontrivial zero-polynomial. In other words, using the polynomial free ideal ring P, which is a zero-polynomial, one can determine the values of the constants (i.e., a_{ij}, b_{ij}, α, β, etc.) within the set of [truncated] Laurent series and its formal power series. Ultimately, (s)he should be able to derive at least one exact solution u_e for the differential equation F.

Lemma 3.8. *If an exact solution u_e exists for the differential equation F, then the differential equation F vanishes when the exact solution u_e is plugged into the equation.*

Once an exact solution u_e is placed into the differential equation/polynomial ring F, an individual obtains zero. In other words, the differential equation F becomes a zero-polynomial ring, like the free ideal ring P, after the introduction of the general solutions u_g with solved constants (i.e., coefficients/parameters, etc.).

Theorem 3.9. *If one is dealing with a [homogeneous] partial differential equation F, which occurs in the physical universe, then (s)he can utilize a set of Laurent series of formal power series, comprised of combinatorial numbers (specifically Fibonacci and Chebyshev numbers about zero)/trigonometric-based parameters and some function f (which is the solution to an ordinary differential equation), to find exact solutions u_e to the equation F.*

Besides expounding upon the theorem developed by Stone-Weierstrass, this theorem is analogous to, but not the same as the Cauchy-Kovalevskaya [7, 21, 22]. Both theorems suggest that if a (system of) equation[s] is analytical, then the solution[s] will be analytical. However, this new theorem does not require Cauchy initial or other conditions (i.e., Neumann, Dirichlet) for the derivation of exact solutions. The examples shown below will provide proof.

4. Examples

All calculations were performed with Mathematica®. The supplemental to this paper contains Mathematica® spreadsheets for each example. Finally, all transformed general solutions U were based upon polynomial exponential rings.

4.1. The Benjamin-Ono equation

The nonlinear Benjamin-Ono equation is defined as follows:

$$u_t + u_{xx} + uu_x = 0.$$

The transformed [N]PDE F in terms of the transformed solution $U(\xi)$ is:

$$\alpha U_\xi + \beta^2 U_{\xi\xi} + \beta\left(\frac{U^2}{2}\right)_\xi = 0.$$

First, one considers and solves the following first-order linear ODE/auxiliary equation:

$$f'(\xi)+f(\xi)=0.$$

Again, its solution is defined as:

$$f(\xi)=c_1e^{-\xi}.$$

Then one calculates the possible maximal/minimal power of the solution p_s. An individual obtains a p_s equal to *1* for this situation. Then (s)he plugs in the p_s value and the solution *f* to the auxiliary/characteristic equation into the transformed general solution *U(ξ)*. It is important to note the transformed general solution *U(ξ)* satisfies *definitions 3.1* and *3.2*. Next, the individual applies the transformed general solution *U* into the transformed [N]PDE or applies *lemma 3.3*. By multiplying this transformed [N]PDE with its common denominator, or implementing *lemma 3.4*, (s)he produces a large expression, which should be a free ideal ring (*lemma 3.5*). This large expression can be used to establish at most eighteen algebraic equations linked to the generator sets $\langle e^{\xi}\rangle$:

$$\beta a(1,0)^2+2\beta a(2,0)a(1,0)+\beta a(2,0)^2=0,$$

$$3\beta c_1^{17}a(1,0)b(1,0)-3\beta c_1^{17}a(2,0)b(1,0)-$$

$$-3\beta c_1^{17}a(1,0)b(2,0)+3\beta c_1^{17}a(2,0)b(2,0)=0,$$

$$-6\beta c_1^{14}b(1,0)^2-6\beta c_1^{14}b(2,0)^2+12\beta c_1^{14}b(1,0)b(2,0)=0,$$

$$2\beta c_1^{18}b(1,0)^2+2\beta c_1^{18}b(2,0)^2-4\beta c_1^{18}b(1,0)b(2,0)=0,$$

$$2\beta c_1^{15}a(1,0)b(1,0)+4\beta c_1^{15}a(1,2)b(1,0)-4\beta c_1^{15}a(2,2)b(1,0)+$$

$$+2\beta c_1^{15}a(1,0)b(1,1)-2\beta c_1^{15}a(2,0)b(1,1)-4\beta c_1^{15}a(1,2)b(2,0)-$$

$$-2\beta c_1^{15}a(2,0)b(2,0)+4\beta c_1^{15}a(2,2)b(2,0)+2\beta c_1^{15}a(1,0)b(2,1)-$$

$$-2\beta c_1^{15}a(2,0)b(2,1)+2\alpha c_1^{15}a(1,0)-2\alpha c_1^{15}a(2,0)-2\beta^2c_1^{15}a(1,0)+$$

$$+2\beta^2c_1^{15}a(2,0)+2\beta c_1^{15}a(1,0)a(1,1)-2\beta c_1^{15}a(1,1)a(2,0)+$$

$$+2\beta c_1^{15}a(1,0)a(2,1)-2\beta c_1^{15}a(2,0)a(2,1)=0,$$

$$4\beta c_1^{16}a(1,1)b(1,0)+4\beta c_1^{16}a(2,1)b(1,0)-4\beta c_1^{16}a(1,1)b(2,0)-$$

$$-4\beta c_1^{16}a(2,1)b(2,0)+\beta c_1^{16}a(1,0)^2+\beta c_1^{16}a(2,0)^2-$$

$$-2\beta c_1^{16}a(1,0)a(2,0)+4\alpha c_1^{16}b(1,0)-4\alpha c_1^{16}b(2,0)-8\beta^2c_1^{16}b(1,0)+$$

$$+8\beta^2c_1^{16}b(2,0)+2\beta c_1^{16}b(1,0)^2-2\beta c_1^{16}b(2,0)^2+4\beta c_1^{16}b(1,0)b(1,1)-$$

$$-4\beta c_1^{16}b(1,1)b(2,0)+4\beta c_1^{16}b(1,0)b(2,1)-4\beta c_1^{16}b(2,0)b(2,1)=0,$$

$$\beta c_1 a(1,0)b(1,0)+\beta c_1 a(2,0)b(1,0)+2\beta c_1 a(1,0)b(1,1)+2\beta c_1 a(2,0)b(1,1)+$$

$$+4\beta c_1 a(1,0)b(1,2)+4\beta c_1 a(2,0)b(1,2)+\beta c_1 a(1,0)b(2,0)+\beta c_1 a(2,0)b(2,0)+$$

$$+2\beta c_1 a(1,0)b(2,1)+2\beta c_1 a(2,0)b(2,1)+4\beta c_1 a(1,0)b(2,2)+4\beta c_1 a(2,0)b(2,2)+$$

$$+2\alpha c_1 a(1,0)+2\alpha c_1 a(2,0)+2\beta^2 c_1 a(1,0)+2\beta^2 c_1 a(2,0)+2\beta c_1 a(1,0)a(1,1)+$$

$$+2\beta c_1 a(1,1)a(2,0)+2\beta c_1 a(1,0)a(2,1)+2\beta c_1 a(2,0)a(2,1)=0,$$

$$-10\beta c_1^{13}a(1,0)b(1,0)+8\beta c_1^{13}a(2,0)b(1,0)+8\beta c_1^{13}a(2,2)b(1,0)-$$

$$-2\beta c_1^{13}a(1,0)b(1,1)-8\beta c_1^{13}a(1,2)b(1,1)-2\beta c_1^{13}a(2,0)b(1,1)+8\beta c_1^{13}a(2,2)b(1,1)-$$

$$-4\beta c_1^{13}a(1,0)b(1,2)+4\beta c_1^{13}a(2,0)b(1,2)+8\beta c_1^{13}a(1,0)b(2,0)-$$

$$-8\beta c_1^{13}a(1,2)b(2,0)-10\beta c_1^{13}a(2,0)b(2,0)-2\beta c_1^{13}a(1,0)b(2,1)-$$

$$-8\beta c_1^{13}a(1,2)b(2,1)-2\beta c_1^{13}a(2,0)b(2,1)+8\beta c_1^{13}a(2,2)b(2,1)+$$

$$+4\beta c_1^{13}a(1,0)b(2,2)-4\beta c_1^{13}a(2,0)b(2,2)-2\alpha c_1^{13}a(1,0)-8\alpha c_1^{13}a(1,2)-$$

$$-2\alpha c_1^{13}a(2,0)+8\alpha c_1^{13}a(2,2)-2\beta^2 c_1^{13}a(1,0)-8\beta^2 c_1^{13}a(1,2)-2\beta^2 c_1^{13}a(2,0)+$$

$$+8\beta^2 c_1^{13}a(2,2)-2\beta c_1^{13}a(1,0)a(1,1)-8\beta c_1^{13}a(1,1)a(1,2)-2\beta c_1^{13}a(1,1)a(2,0)-$$

$$-2\beta c_1^{13}a(1,0)a(2,1)-8\beta c_1^{13}a(1,2)a(2,1)-2\beta c_1^{13}a(2,0)(2,1)+$$

$$+8\beta c_1^{13}a(1,1)a(2,2)+8\beta c_1^{13}a(2,1)a(2,2)=0,$$

$$-2\beta c_1^{3}a(1,0)b(1,0)-4\beta c_1^{3}a(1,2)b(1,0)-4\beta c_1^{3}a(2,2)b(1,0)-2\beta c_1^{3}a(1,0)b(1,1)-$$

$$-8\beta c_1^{3}a(1,2)b(1,1)+2\beta c_1^{3}a(2,0)b(1,1)-8\beta c_1^{3}a(2,2)b(1,1)-16\beta c_1^{3}a(1,2)b(1,2)+$$

$$+8\beta c_1^{3}a(2,0)b(1,2)-16\beta c_1^{3}a(2,2)b(1,2)-4\beta c_1^{3}a(1,2)b(2,0)+$$

$$+2\beta c_1^{3}a(2,0)b(2,0)-4\beta c_1^{3}a(2,2)b(2,0)-2\beta c_1^{3}a(1,0)b(2,1)-8\beta c_1^{3}a(1,2)b(2,1)+$$

$$+2\beta c_1^{3}a(2,0)b(2,1)-8\beta c_1^{3}a(2,2)b(2,1)-8\beta c_1^{3}a(1,0)b(2,2)-$$

$$-16\beta c_1^{3}a(1,2)b(2,2)-16\beta c_1^{3}a(2,2)b(2,2)-2\alpha c_1^{3}a(1,0)-8\alpha c_1^{3}a(1,2)+$$

$$+2\alpha c_1^{3}a(2,0)-8\alpha c_1^{3}a(2,2)+2\beta^2 c_1^{3}a(1,0)+8\beta^2 c_1^{3}a(1,2)-2\beta^2 c_1^{3}a(2,0)+$$

$$+8\beta^2 c_1^{3}a(2,2)-2\beta c_1^{3}a(1,0)a(1,1)-8\beta c_1^{3}a(1,1)a(1,2)+2\beta c_1^{3}a(1,1)a(2,0)-$$

$$-2\beta c_1^3 a(1,0)a(2,1)-8\beta c_1^3 a(1,2)a(2,1)+2\beta c_1^3 a(2,0)a(2,1)-$$

$$-8\beta c_1^3 a(1,1)a(2,2)-8\beta c_1^3 a(2,1)a(2,2)=0,$$

$$-6\beta c_1^5 a(1,0)b(1,0)-24\beta c_1^5 a(2,2)b(1,0)-6\beta c_1^5 a(1,0)b(1,1)+24\beta c_1^5 a(1,2)b(1,1)-$$

$$-6\beta c_1^5 a(2,0)b(1,1)-24\beta c_1^5 a(2,2)b(1,1)-12\beta c_1^5 a(1,0)b(1,2)+$$

$$+96\beta c_1^5 a(1,2)b(1,2)-36\beta c_1^5 a(2,0)b(1,2)+24\beta c_1^5 a(1,2)b(2,0)-$$

$$-6\beta c_1^5 a(2,0)b(2,0)-6\beta c_1^5 a(1,0)b(2,1)+24\beta c_1^5 a(1,2)b(2,1)-6\beta c_1^5 a(2,0)b(2,1)-$$

$$-24\beta c_1^5 a(2,2)b(2,1)-36\beta c_1^5 a(1,0)b(2,2)-12\beta c_1^5 a(2,0)b(2,2)-$$

$$-96\beta c_1^5 a(2,2)b(2,2)-6\alpha c_1^5 a(1,0)+24\alpha c_1^5 a(1,2)-6\alpha c_1^5 a(2,0)-24\alpha c_1^5 a(2,2)-$$

$$-6\beta^2 c_1^5 a(1,0)-72\beta^2 c_1^5 a(1,2)-6\beta^2 c_1^5 a(2,0)+72\beta^2 c_1^5 a(2,2)-6\beta c_1^5 a(1,0)a(1,1)+$$

$$+24\beta c_1^5 a(1,1)a(1,2)-6\beta c_1^5 a(1,1)a(2,0)-6\beta c_1^5 a(1,0)a(2,1)+$$

$$+24\beta c_1^5 a(1,2)a(2,1)-6\beta c_1^5 a(2,0)a(2,1)-$$

$$-24\beta c_1^5 a(1,1)a(2,2)-24\beta c_1^5 a(2,1)a(2,2)=0,$$

$$6\beta c_1^7 a(1,0)b(1,0)+12\beta c_1^7 a(1,2)b(1,0)-28\beta c_1^7 a(2,2)b(1,0)+6\beta c_1^7 a(1,0)b(1,1)-$$

$$-16\beta c_1^7 a(1,2)b(1,1)-6\beta c_1^7 a(2,0)b(1,1)-16\beta c_1^7 a(2,2)b(1,1)-$$

$$-192\beta c_1^7 a(1,2)b(1,2)+16\beta c_1^7 a(2,0)b(1,2)-32\beta c_1^7 a(2,2)b(1,2)-$$

$$-28\beta c_1^7 a(1,2)b(2,0)-6\beta c_1^7 a(2,0)b(2,0)+12\beta c_1^7 a(2,2)b(2,0)+$$

$$+6\beta c_1^7 a(1,0)b(2,1)-16\beta c_1^7 a(1,2)b(2,1)-6\beta c_1^7 a(2,0)b(2,1)-16\beta c_1^7 a(2,2)b(2,1)-$$

$$-16\beta c_1^7 a(1,0)b(2,2)-32\beta c_1^7 a(1,2)b(2,2)-192\beta c_1^7 a(2,2)b(2,2)+$$

$$+6\alpha c_1^7 a(1,0)-16\alpha c_1^7 a(1,2)-6\alpha c_1^7 a(2,0)-16\alpha c_1^7 a(2,2)-6\beta^2 c_1^7 a(1,0)+$$

$$+176\beta^2 c_1^7 a(1,2)+6\beta^2 c_1^7 a(2,0)+176\beta^2 c_1^7 a(2,2)+6\beta c_1^7 a(1,0)a(1,1)-$$

$$-16\beta c_1 a(1,1)a(1,2)-6\beta c_1^7 a(1,1)a(2,0)+6\beta c_1^7 a(1,0)a(2,1)-16\beta c_1^7 a(1,2)a(2,1)-$$

$$-6\beta c_1^7 a(2,0)a(2,1)-16\beta c_1^7 a(1,1)a(2,2)-16\beta c_1^7 a(2,1)a(2,2)=0,$$

$$12\beta c_1^9 a(1,0)b(1,0)-6\beta c_1^9 a(2,0)b(1,0)+16\beta c_1^9 a(2,2)b(1,0)+6\beta c_1^9 a(1,0)b(1,1)-$$

$$-16\beta c_1^9 a(1,2)b(1,1)+6\beta c_1^9 a(2,0)b(1,1)+16\beta c_1^9 a(2,2)b(1,1)+$$

$$+12\beta c_1^9 a(1,0)b(1,2)+160\beta c_1^9 a(1,2)b(1,2)+28\beta c_1^9 a(2,0)b(1,2)-$$

$$-6\beta c_1^9 a(1,0)b(2,0)-16\beta c_1^9 a(1,2)b(2,0)+12\beta c_1^9 a(2,0)b(2,0)+$$

$$+6\beta c_1^9 a(1,0)b(2,1)-16\beta c_1^9 a(1,2)b(2,1)+6\beta c_1^9 a(2,0)b(2,1)+$$

$$+16\beta c_1^9 a(2,2)b(2,1)+28\beta c_1^9 a(1,0)b(2,2)+12\beta c_1^9 a(2,0)b(2,2)-$$

$$-160\beta c_1^9 a(2,2)b(2,2)+6\alpha c_1^9 a(1,0)-16\alpha c_1^9 a(1,2)+6\alpha c_1^9 a(2,0)+$$

$$+16\alpha c_1^9 a(2,2)+6\beta^2 c_1^9 a(1,0)-176\beta^2 c_1^9 a(1,2)+6\beta^2 c_1^9 a(2,0)+176\beta^2 c_1^9 a(2,2)+$$

$$+6\beta c_1^9 a(1,0)a(1,1)-16\beta c_1^9 a(1,1)a(1,2)+6\beta c_1^9 a(1,1)a(2,0)+$$

$$+6\beta c_1^9 a(1,0)a(2,1)-16\beta c_1^9 a(1,2)a(2,1)+6\beta c_1^9 a(2,0)a(2,1)+$$

$$+16\beta c_1^9 a(1,1)a(2,2)+16\beta c_1^9 a(2,1)a(2,2)=0,$$

$$-6\beta c_1^{11} a(1,0)b(1,0)-12\beta c_1^{11} a(1,2)b(1,0)+36\beta c_1^{11} a(2,2)b(1,0)-$$

$$-6\beta c_1^{11} a(1,0)b(1,1)+24\beta c_1^{11} a(1,2)b(1,1)+36\beta c_1^{11} a(1,2)b(2,0)+$$

$$+6\beta c_1^{11} a(2,0)b(2,0)-12\beta c_1^{11} a(2,2)b(2,0)-6\beta c_1^{11} a(1,0)b(2,1)+$$

$$+24\beta c_1^{11} a(1,2)b(2,1)+6\beta c_1^{11} a(2,0)b(2,1)+24\beta c_1^{11} a(2,2)b(2,1)+$$

$$+24\beta c_1^{11} a(1,0)b(2,2)+48\beta c_1^{11} a(1,2)b(2,2)-48\beta c_1^{11} a(2,2)b(2,2)-$$

$$-6\alpha c_1^{11} a(1,0)+24\alpha c_1^{11} a(1,2)+6\alpha c_1^{11} a(2,0)+24\alpha c_1^{11} a(2,2)+$$

$$+6\beta^2 c_1^{11} a(1,0)+72\beta^2 c_1^{11} a(1,2)-6\beta^2 c_1^{11} a(2,0)+72\beta^2 c_1^{11} a(2,2)-$$

$$-6\beta c_1^{11} a(1,0)a(1,1)+24\beta c_1^{11} a(1,1)a(1,2)+6\beta c_1^{11} a(1,1)a(2,0)-$$

$$-6\beta c_1^{11} a(1,0)a(2,1)+24\beta c_1^{11} a(1,2)a(2,1)+6\beta c_1^{11} a(2,0)a(2,1)+$$

$$+24\beta c_1^{11} a(1,1)a(2,2)+24\beta c_1^{11} a(2,1)a(2,2)=0,$$

$$-12\beta c_1^{12} a(1,1)b(1,0)-12\beta c_1^{12} a(2,1)b(1,0)-16\beta c_1^{12} a(1,1)b(1,2)-$$

$$-16\beta c_1^{12} a(2,1)b(1,2)+12\beta c_1^{12} a(1,1)b(2,0)+12\beta c_1^{12} a(2,1)b(2,0)+$$

$$+16\beta c_1^{12} a(1,1)b(2,2)+16\beta c_1^{12} a(2,1)b(2,2)-4\beta c_1^{12} a(1,0)^2-16\beta c_1^{12} a(1,2)^2-$$

$$-4\beta c_1^{12} a(2,0)^2-16\beta c_1^{2} a(2,2)^2+4\beta c_1^{12} a(1,0)a(2,0)-16\beta c_1^{12} a(1,2)a(2,0)+$$

$$+16\beta c_1^{12} a(1,0)a(2,2)+32\beta c_1^{12} a(1,2)a(2,2)-12\alpha c_1^{12} b(1,0)-16\alpha c_1^{12} b(1,2)+$$

$$+12\alpha c_1^{12}b(2,0)+16\alpha c_1^{12}b(2,2)+24\beta^2c_1^{12}b(1,0)-32\beta^2c_1^{12}b(1,2)-24\beta^2c_1^{12}b(2,0)+$$

$$+32\beta^2c_1^{12}b(2,2)-6\beta c_1^{12}b(1,0)^2+6\beta c1^{12}b(2,0)^2-12\beta c_1^{12}b(1,0)b(1,1)-$$

$$-16\beta c_1^{12}b(1,1)b(1,2)+12\beta c_1^{12}b(1,1)b(2,0)-16\beta c_1^{12}b(1,2)b(2,0)-$$

$$-12\beta c_1^{12}b(1,0)b(2,1)-16\beta c_1^{12}b(1,2)b(2,1)+12\beta c_1^{12}b(2,0)b(2,1)+$$

$$+16\beta c_1^{12}b(1,0)b(2,2)+16\beta c_1^{12}b(1,1)b(2,2)+16\beta c_1^{12}b(2,1)b(2,2)=0,$$

$$-4\beta c_1^4a(1,1)b(1,0)-4\beta c_1^4a(2,1)b(1,0)+16\beta c_1^4a(1,1)b(1,2)+16\beta c_1^4a(2,1)b(1,2)+$$

$$+4\beta c_1^4a(1,1)b(2,0)-4\beta c_1^4a(2,1)b(2,0)-16\beta c_1^4a(1,1)b(2,2)-$$

$$-16\beta c_1^4a(2,1)b(2,2)-4\beta c_1^4a(1,0)^2-16\beta c_1^4a(1,2)^2-4\beta c_1^4a(2,0)^2-16\beta c_1^4a(2,2)^2-$$

$$-4\beta c_1^4a(1,0)a(2,0)+16\beta c_1^4a(1,2)a(2,0)-16\beta c_1^4a(1,0)a(2,2)-$$

$$-32\beta c_1^4a(1,2)a(2,2)-4\alpha c_1^4b(1,0)+16\alpha c_1^4b(1,2)+4\alpha c_1^4b(2,0)-16\alpha c_1^4b(2,2)+$$

$$+8\beta^2c_1^4b(1,0)-32\beta^2c_1^4b(1,2)-8\beta^2c_1^4b(2,0)+32\beta^2c_1^4b(2,2)-2\beta c_1^4b(1,0)^2+$$

$$+32\beta c_1^4b(1,2)^2+2\beta c_1^4b(2,0)^2-32\beta c_1^4b(2,2)^2-4\beta c_1^4b(1,0)b(1,1)+$$

$$+16\beta c_1^4b(1,1)b(1,2)+4\beta c_1^4b(1,1)b(2,0)+16\beta c_1^4b(1,2)b(2,0)-$$

$$-4\beta c_1^4b(1,0)b(2,1)+16\beta c_1^4b(1,2)b(2,1)+4\beta c_1^4b(2,0)b(2,1)-$$

$$-16\beta c_1^4b(1,0)b(2,2)-16\beta c_1^4b(1,1)b(2,2)-16\beta c_1^4b(2,1)b(2,2)=0,$$

$$-32\beta c_1^6a(1,1)b(1,2)-32\beta c_1^6a(2,1)b(1,2)-32\beta c_1^6a(1,1)b(2,2)-$$

$$-32\beta c_1^6a(2,1)b(2,2)+64\beta c_1^6a(1,2)^2-64\beta c_1^6a(2,2)^2-32\beta c_1^6a(1,2)a(2,0)-$$

$$-32\beta c_1^6a(1,0)a(2,2)-32\alpha c_1^6b(1,2)-32\alpha c_1^6b(2,2)+128\beta^2c_1^6b(1,2)+$$

$$+128\beta^2c_1^6b(2,2)-2\beta c_1^6b(1,0)^2-96\beta c_1^6b(1,2)^2-2\beta c_1^6b(2,0)^2-96\beta c_1^6b(2,2)^2-$$

$$-32\beta c_1^6b(1,1)b(1,2)+4\beta c_1^6b(1,0)b(2,0)-32\beta c_1^6b(1,2)b(2,0)-$$

$$-32\beta c_1^6b(1,2)b(2,1)-32\beta c_1^6b(1,0)b(2,2)-32\beta c_1^6b(1,1)b(2,2)-$$

$$-64\beta c_1^6b(1,2)b(2,2)-32\beta c_1^6b(2,1)b(2,2)=0,$$

$$12\beta c_1^8a(1,1)b(1,0)+12\beta c_1^8a(2,1)b(1,0)-12\beta c_1^8a(1,1)b(2,0)-$$

$$-12\beta c_1^8a(2,1)b(2,0)+6\beta c_1^8a(1,0)^2-96\beta c_1^8a(1,2)^2+6\beta c_1^8a(2,0)^2-$$

$$-96\beta c_1^8 a(2,2)^2+12\alpha c_1^8 b(1,0)-12\alpha c_1^8 b(2,0)-24\beta^2 c_1^8 b(1,0)-192\beta^2 c_1^8 b(1,2)+$$
$$+24\beta^2 c_1^8 b(2,0)+192\beta^2 c_1^8 b(2,2)+6\beta c_1^8 b(1,0)^2+96\beta c_1^8 b(1,2)^2-6\beta c_1^8 b(2,0)^2-$$
$$-96\beta c_1^8 b(2,2)^2+12\beta c_1^8 b(1,0)b(1,1)-12\beta c_1^8 b(1,1)b(2,0)+$$
$$+12\beta c_1^8 b(1,0)b(2,1)-12\beta c_1^8 b(2,0)b(2,1)=0,$$

and

$$32\beta c_1^{10} a(1,1)b(1,2)+32\beta c_1^{10} a(2,1)b(1,2)+32\beta c_1^{10} a(1,1)b(2,2)+$$
$$+32\beta c_1^{10} a(2,1)b(2,2)+64\beta c_1^{10} a(1,2)^2-64\beta c_1^{10} a(2,2)^2+32\beta c_1^{10} a(1,2)a(2,0)+$$
$$+32\beta c_1^{10} a(1,0)a(2,2)+32\alpha c_1^{10} b(1,2)+32\alpha c_1^{10} b(2,2)+128\beta^2 c_1^{10} b(1,2)+$$
$$+128\beta^2 c_1^{10} b(2,2)+6\beta c_1^{10} b(1,0)^2-32\beta c_1^{10} b(1,2)^2+6\beta c_1^{10} b(2,0)^2-32\beta c_1^{10} b(2,2)^2+$$
$$+32\beta c_1^{10} b(1,1)b(1,2)-12\beta c_1^{10} b(1,0)b(2,0)+32\beta c_1^{10} b(1,2)b(2,0)+$$
$$+32\beta c_1^{10} b(1,2)b(2,1)+32\beta c_1^{10} b(1,0)b(2,2)+32\beta c_1^{10} b(1,1)b(2,2)+$$
$$+64\beta c_1^{10} b(1,2)b(2,2)+32\beta c_1^{10} b(2,1)b(2,2)=0.$$

The eighteen algebraic equations are used to solve for constants a_{ij}, b_{ij}, α, β, and c_1 whenever possible via *lemma 3.6*. Substituting in the previously described constants into the transformed general solution gives rise to a set of possible exact solutions u like:

$$u(x,t)=\frac{2\beta e^{\beta x}}{e^{\beta x}-c_1 e^{\beta t(a_{11}+a_{21}+\beta+b_{10}+b_{11}+b_{21})}}+a_{11}+a_{21}+b_{10}+b_{11}+b_{21}.$$

Note: the above solution u satisfies *definition 3.7*. Regarding *lemma 3.8*, plugging the above exact solution u into the original [N]PDE F causes the latter to vanish. Thus, *theorem 3.9* is proven true.

Now an individual can derive new solutions if they change the characteristic/auxiliary equation to a second-order linear ODE given below:

$$f''(\xi)+f'(\xi)+f(\xi)=0.$$

The solution to this equation is as follows:

$$f(\xi)=c_1 e^{-\frac{\xi}{2}}\sin\left(\frac{\sqrt{3}\xi}{2}\right)+c_2 e^{-\frac{\xi}{2}}\cos\left(\frac{\sqrt{3}\xi}{2}\right).$$

Next, the transformed Benjamin-Ono equation can be simplified by integrating it with respect to the transform variable ξ; therefore, one is left with the following [N]PDE:

$$\alpha U+\beta^2 U_\xi+\beta\frac{U^2}{2}=0.$$

Then (s)he plugs in the same p_s value and the solution f above into the transformed general solution $U(\xi)$; therefore, satisfying *definitions 3.1* and *3.2*. Next, the individual applies the transformed solution U into the transformed [N]PDE F or implements *lemma 3.3*. By multiplying this transformed [N]PDE F with its common denominator, (s)he produces a large expression (*lemma 3.4*) that establishes at most forty-five algebraic equations linked to generators $\langle e^{\frac{\xi}{2}}, e^{\frac{i\sqrt{3}\xi}{2}}\rangle$ or *lemma 3.5*. (The free ideal ring P for this example has a rank of *2*.) The forty-five algebraic equations are used to solve for constants a_{ij}, b_{ij}, α, and β whenever possible via *lemma 3.6*. Substituting in the previously described constants into the transformed general solution gives rise to the final exact solution[s] u like:

$$u(x,t)=\frac{2i\left(\sqrt{3}-i\right)\beta e^{\beta(2\beta t+x)}}{e^{\beta(2\beta t+x)}-ic_2^2 e^{2i\sqrt{3}\beta^2 t}\sin\left(\sqrt{3}\beta x\right)+c_2^2 e^{2i\sqrt{3}\beta^2 t}\cos\left(\sqrt{3}\beta x\right)}.$$

The latter solution to the equation is new or exotic and satisfies *definition 3.7*. Regarding *lemma 3.8,* the exact solution u causes [N]PDE F to vanish. Again, *theorem 3.9* is proven true.

4.2. A *QFT (nonlinear Klein-Gordon)[-like] equation.*

The QFT[-like] equation is defined as the following expression:

$$u_{tt}+u_{xx}+u+u^3=0.$$

The transformed [N]PDE F in terms of the transformed solution $U(\xi)$ is:

$$\alpha^2 U_{\xi\xi}+\beta^2 U_{\xi\xi}+U+U^3=0.$$

First, one calculates the possible maximal/minimal power of the solution p_s. An individual obtains a p_s equal to *1* for this situation. Then (s)he determines the solution to the following auxiliary equation:

$$f''(\xi)+f(\xi)=0,$$

which is:

$$f(\xi)=c_1\cos(\xi)+c_2\sin(\xi).$$

Next, (s)he plugs in the p_s value and the solution f to the characteristic/auxiliary equation into the transformed general solution $U(\xi)$ thereby satisfying *definitions 3.1* and *3.2*. Next, the individual applies the transformed solution into the transformed [N]PDE F or applies *lemma 3.3*. By multiplying this transformed [N]PDE with its common denominator, (s)he produces a large expression (*lemma 3.4*) which establishes at most thirteen algebraic equations linked to the set of generators $\langle e^{i\xi}\rangle$ or *lemma 3.5*. The thirteen algebraic equations are used to solve for constants a_{ij}, b_{ij}, α, and β whenever possible through *lemma 3.6*. Substituting in the previously described constants into the transformed general solution gives rise to the final exact solution[s] u like:

$$u(x,t)=\frac{i\sqrt{2}}{\sqrt{-c_1^2-2c_1\sin\left(2at-\sqrt{1-4a^2}x\right)+\left(c_1^2+1\right)\cos\left(2at-\sqrt{1-4a^2}x\right)}}$$

and

$$u(x,t)=2i\cdot\left[-c_2^2\sin^2\left(\alpha t+\sqrt{-\alpha^2-\frac{1}{2}x}\right)-ic_2^2\sin\left(2at+\sqrt{-4\alpha^2-2\frac{1}{2}x}\right)+\right.$$

$$\left.+c_2^2\cos^2(\alpha t+\sqrt{-\alpha^2-\frac{1}{2}x)-1}\right]^{-1}+i.$$

The above solutions are considered new or exotic and satisfy *definition 3.7*. Apropos *lemma 3.8*, the exact solutions cause the [N]PDE F to vanish and ultimately prove *theorem 3.9*.

4.3. *The Good Boussinesq[-like] equation.*

The Good Boussinesq[-like] equation is defined as the following expression:

$$u_{tt}+u_{xx}+u_{xxxx}+\left(\frac{u^2}{2}\right)_{xx}=0.$$

The transformed [N]PDE F in terms of the transformed solution $U(\xi)$ is:

$$\alpha^2U_{\xi\xi}+\beta^2U_{\xi\xi}+\beta^4U_{\xi\xi\xi\xi}+\beta^2\left(\frac{U^2}{2}\right)_{\xi\xi}=0.$$

This equation can be simplified by integrating it with respect to the transform variable ξ; therefore, one is left with the following [N]PDE:

$$\alpha^2U+\beta^2U+\beta^4U_{\xi\xi}+\beta^2\left(U^2/2\right)=0.$$

First, one calculates the possible maximal/minimal power of the solution p_s. An individual would obtain a p_s equal to *2* for this situation. Next (s)he finds a solution to the auxiliary/characteristic equation, which is a linear ODE, like:

$$f^{(3)}(\xi)+f(\xi)=0.$$

The above equation solution is:

$$f(\xi)=c_1e^{-\xi}+c_2e^{\frac{\xi}{2}}\sin\left(\frac{\sqrt{3}\xi}{2}\right)+c_3e^{\frac{\xi}{2}}\cos\left(\frac{\sqrt{3}\xi}{2}\right).$$

Then the individual plugs in the p_s value and the solution f of the characteristic/auxiliary equation into the transformed general solution $U(\xi)$ that satisfies *definitions 3.1* and *3.2*. Next, the individual applies the transformed solution U into the transformed [N]PDE F or *lemma 3.3*. By multiplying this transformed [N]PDE with its common denominator, (s)he produces a large expression (*lemma 3.4*) which establishes at most four-hundred and sixty-five algebraic equations linked to generator set $e^{\frac{\xi}{2}}, e^{\frac{i\sqrt{3}\xi}{2}}$ or *lemma 3.5*. (Even though one would anticipate the rank for this free ideal ring P to be *2*, (s)he obtains a rank of *7* for the algebraic structure.) These algebraic equations are used to solve for constants a_{ij}, b_{ij}, α, and β whenever possible via *lemma 3.6*. Substituting in the previously described constants into the transformed general solution gives rise to the final exact solution[s] u like:

$$u(x,t) = \frac{96\left(1 - i\sqrt{3}\right)\beta^2 c_2^4 e^{2\xi}}{\left(\left(c_2^4 e^{2\xi} - 1\right)\cos\left(\sqrt{3}\xi\right) + i\left(c_2^4 e^{2\xi} + 1\right)\sin\left(\sqrt{3}\xi\right)\right)^2}$$

where

$$\xi = \beta x + \frac{\sqrt{16\sqrt{3}\beta^4 - 16i\beta^4 - \sqrt{3}\beta^2 - i\beta^2}\,t}{\sqrt{\sqrt{3} + i}}.$$

The above solution is considered new or exotic and satisfies *definition 3.7*. The exact solution u makes the [N]PDE F vanish as defined by *lemma 3.8*. Therefore, *theorem 3.9* is proven true.

Conclusion

The distinction between SEM, the G'/G-expansion, and GFT

The major difference between SEM, the G'/G-expansion method, and GFT is how their auxiliary equations are used. The auxiliary equation utilized in SEM create the template by which solutions are established while the generating function performs that same task for GFT. The auxiliary equation usage in GFT is for adding complexity or greater diversification to the template through which solutions are established. The auxiliary equation is used in similar fashion by the G'/G-expansion method but to a much lesser extent. In other words, the G'/G-expansion method's auxiliary equation usage is a lot more limited with respect to GFT. As discussed before, an auxiliary equation that is a first-order linear ODE will let GFT create solutions like SEM. It is important to note these solutions tend to be primarily comprised of a [hyperbolic] secant, cosecant, tangent, or cotangent function in the numerator position. If an individual uses a higher-order auxiliary equation, (s)he produces a greater variety of solutions where "differing (and large) combinations" of [hyperbolic] sine, cosine, and exponential functions appear in both the numerator and denominator positions.

From one dimension to beyond

GFT can be used to solve a large range of PDEs including problems that have more than one spatial dimension. This paper primarily focuses on the generation of soliton-based solutions for (1 + 1) PDEs; thus, the "bilinear" form of GFT is only utilized in this study. If one needs to solve (N + 1) PDEs, where N $\geq$ 2, then the individual just adds more coefficients and variables or intermediates to ξ, like the following for N = 3:

$$\xi = \alpha t + \beta_1 x + \beta_2 y + \beta_3 z,$$

then make the appropriate transformations to the PDE. Next, (s)he can find the exact solution by committing the same steps used to solve 1 + 1 equations, but one must also solve for additional coefficients of the added variables or intermediates if deemed necessary. (Generally, (s)he just needs to solve for α concerning the other coefficients.) Therefore, "multilinear" GFT can solve N + 1 PDEs. An individual can also apply "unilinear" GFT to solve ordinary differential equations, by restricting ξ to one specific coefficient and variable/intermediate product, like αt, then committing the same steps that are described above. Finally, GFT can be used to generate new and exotic solutions to regular linear [partial] differential equations if one uses a p_s equal to or lesser than *-1*.

References

1. Lindenstrauss J., Evans L.C., Douady A., Shalev A., Pippenger N. *Fields Medals and Nevanlinna Prize presented at ICM-94 in Zürich, Notices Amer. Math. Soc*. 1994;41(9):1103–1111.
2. Adomian G. *Solving Frontier Problems of Physics: The decomposition method*. Kluwer Academic Publishers, 1994.
3. Liao S.J. *Homotopy Analysis Method in Nonlinear Differential Equation*. Berlin & Beijing: Springer & Higher Education Press, 2012.
4. Kudryashov N. Exact solitary waves of the Fisher equation. *Physics Letters A*. 2005;342(1-2):99–106.
5. He J.H. Exp-function method for nonlinear wave equations. *Chaos, Solitons and Fractals*. 2006;30:700–708
6. Wang M. The (G′/G)-expansion method and traveling wave solutions of nonlinear evolution equations in mathematical physics. *Phys Lett A*. 2008;372:417–423.
7. de Branges L. The Stone–Weierstrass theorem. *Proc. Amer. Math. Soc*. 1959;10:822–824. doi:10.1090/s0002-9939-1959-0113131-7
8. Nakhushev A.M. Cauchy–Kovalevskaya theorem. *Encyclopedia of Mathematics, Springer Science+Business Media B. V.* Hazewinkel, Michiel (ed.). Kluwer Academic Publishers, 2001.
9. Chadwick E. Exponential function method for solving nonlinear ordinary differential equations with constant coefficients on a semi-infinite domain. *Proc. Indian Acad. Sci. (Math. Sci.)*. 2015;126(1):79–97.
10. Allenby R.B.J.T. Edward Arnold. *Rings, Fields and Groups (Second ed.)*. London, 1991:xxvi+383.
11. Niven I. Formal Power Series. *American Mathematical Monthly*. 1969;76(8):871–889. doi:10.1080/00029890.1969.12000359
12. Knuth D.E. *The Art of Computer Programming, Volume 1 Fundamental Algorithms (Third Edition)*. Addison-Wesley, 1998.
13. Riccati J. *Animadversiones in aequationes differentiales secundi gradus (Observations regarding differential equations of the second order). Actorum Eruditorum, quae Lipsiae publicantur, Supplementa*. 1724;8:66–73 [Translation of the original Latin into English by Ian Bruce].
14. Rodriguez. *Complex Analysis: In the Spirit of Lipman Bers, Graduate Texts in Mathematics*. Springer, 2012:12.

15. Herstein H. *Topics in Algebra*. Wiley, 1975;Section 3.9;Section 3.6.
16. Dries L. van de. Exponential Rings, Exponential Polynomials and Exponential Functions. *Pacific Journal of Mathematics*. 1984;133(1):51–66.
17. Kobayashi N. *Foundations of Differential Geometry*. Wiley-Interscience, 2004;2.
18. Goodearl K.R., Warfield R.B. *An Introduction to Noncommutative Noetherian Rings. Second Edition. London Mathematical Society Student Texts*. Cambridge: Cambridge University Press, 2004.
19. Gradshteyn I.S., Ryzhik I.M., Geronimus Y.V., Tseytlin M.Yu., Jeffrey A. Table of Integrals, Series, and Products. *Academic Press*, 2015:18.
20. Cohn P.M. Free ideal rings and free products of rings. *Actes du Congrès International des Mathématicien*. Gauthier-Villars, 1971:273–278.
21. Cauchy A. Mémoire sur l'emploi du calcul des limites dans l'intégration des équations aux dérivées partielles. *Comptes rendus, 15 Reprinted in Oeuvres completes, 1 serie*. 1842;VII:17–58.
22. Kowalevsky S. Zur Theorie der partiellen Differentialgleichung. *Journal für die reine und angewandte Mathematik*. 1875;80:1–32.

Список литературы

1. Lindenstrauss J., Evans L. C., Douady A., Shalev A., Pippenger N., Fields Medals and Nevanlinna Prize presented at ICM-94 in Zürich, Notices Amer. Math. Soc. 1994. Vol. 41 (9). R. 1103–1111.
2. Adomian G. Solving Frontier Problems of Physics: The decomposition method. Kluwer Academic Publishers, 1994.
3. Liao S. J. Homotopy Analysis Method in Nonlinear Differential Equation. Berlin & Bei-jing : Springer & Higher Education Press, 2012.
4. Kudryashov N. Exact solitary waves of the Fisher equation // Physics Letters A. 2005. Vol. 342, № 1-2. R. 99–106.
5. He J. H. Exp-function method for nonlinear wave equations // Chaos, Solitons and Fractals. 2006. Vol. 30. R. 700–708
6. Wang M. The (G′/G)-expansion method and traveling wave solutions of nonlinear evolution equations in mathematical physics // Phys Lett A. 2008. Vol. 372. R. 417–423.
7. de Branges L. The Stone–Weierstrass theorem // Proc. Amer. Math. Soc. 1959. Vol. 10. P. 822–824. doi:10.1090/s0002-9939-1959-0113131-7
8. Nakhushev A. M. Cauchy–Kovalevskaya theorem // Encyclopedia of Mathematics, Springer Science+Business Media B. V. / Hazewinkel, Michiel (ed.). Kluwer Academic Publishers, 2001.
9. Chadwick E. Exponential function method for solving nonlinear ordinary differential equations with constant coefficients on a semi-infinite domain // Proc. Indian Acad. Sci. (Math. Sci.). 2015. Vol. 126, № 1. S. 79–97.
10. Allenby, R. B. J. T. Edward Arnold / Rings, Fields and Groups (Second ed.). London, 1991. P. xxvi+383.
11. Niven I. Formal Power Series // American Mathematical Monthly. 1969. Vol. 76, № 8. R. 871–889. doi:10.1080/00029890.1969.12000359
12. Knuth D. E. The Art of Computer Programming, Volume 1 Fundamental Algorithms (Third Edition). Addison-Wesley,1998.
13. Riccati J. Animadversiones in aequationes differentiales secundi gradus (Observations regarding differential equations of the second order). Actorum Eruditorum, quae Lipsiae publicantur, Supplementa. 1724. Vol. 8. R. 66–73 [Translation of the original Latin into English by Ian Bruce].
14. Rodriguez. Complex Analysis: In the Spirit of Lipman Bers, Graduate Texts in Mathematics. Springer, 2012. p. 12.
15. Herstein H. Topics in Algebra. Wiley, 1975. Section 3.9; Section 3.6.

16. Dries L. van de. Exponential Rings, Exponential Polynomials and Exponential Functions // Pacific Journal of Mathematics. 1984. Vol. 133, № 1. R. 51–66.
17. Kobayashi N. Foundations of Differential Geometry. Wiley-Interscience, 2004. Vol. 2.
18. Goodearl K. R., Warfield R. B. An Introduction to Noncommutative Noetherian Rings. Second Edition. London Mathematical Society Student Texts. Cambridge : Cambridge University Press, 2004.
19. Gradshteyn I. S., Ryzhik I. M., Geronimus Y. V., Tseytlin M. Yu., Jeffrey A. Table of In-tegrals, Series, and Products. // Academic Press, 2015. p. 18.
20. Cohn P. M. Free ideal rings and free products of rings // Actes du Congrès International des Mathématicien. Gauthier-Villars, 1971. R. 273–278.
21. Cauchy A. Mémoire sur l'emploi du calcul des limites dans l'intégration des équations aux dérivées partielles // Comptes rendus, 15 Reprinted in Oeuvres completes, 1 serie. 1842. T. VII. R. 17–58.
22. Kowalevsky S. Zur Theorie der partiellen Differentialgleichung // Journal für die reine und angewandte Mathematik. 1875. Vol. 80. R. 1–32.

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